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A note on Artinian Gorenstein algebras of codimension three

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Abstract

In this paper, using a standard fact in linkage theory, we give a new construction of Artinian Gorenstein algebras achieving all possible sets of graded Betti numbers for codimension three. Furthermore, as an application, we give another proof of Stanley's well-known characterization theorem for the Hilbert functions of codimension three Artinian Gorenstein algebras. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

There are some well-known results on the problem of constructing Artinian Gorenstein algebras having an assigned set of graded Betti numbers which are possible for some Artinian Gorenstein algebra of codimension three. We recall some of these results. An explicit construction can be found in the paper by Herzog et al. [8, p. 63] and the paper by Diesel [3, Proposition 3.1]. Furthermore, recently, Geramita and Migliore [5, Theorem 2.1] showed that any admissible set of graded Betti numbers in fact occur for a reduced set of points in $\mathbf{P}^3$. The starting point of these constructions is the well-known structure theorem of Buchsbaum and Eisenbud [1, Theorem 2.1] for Gorenstein ideals of height three.

It is a standard fact in linkage theory [10, Remarque 1.4] that the sum of two geometrically linked Cohen–Macaulay ideals is a Gorenstein ideal of codimension one greater. In this paper, using this idea, we give a new construction of Artinian Gorenstein algebras achieving all possible sets of graded Betti numbers for codimension three. That is, such Artinian Gorenstein algebras can be obtained as the sum of the

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ideals of two geometrically linked finite sets of points in $\mathbf{P}^2$ (Theorem 3.3). Furthermore, as an application of this construction, we show that for any Artinian Gorenstein algebra of codimension three, there exists an Artinian Gorenstein algebra with the weak Stanley property which has the same Hilbert function (Theorem 4.3 and Remark 4.6). Consequently, we give another proof of the well-known theorem of Stanley [11, Theorem 4.2] which gave a characterization of the Hilbert functions of codimension three Artinian Gorenstein algebras (Theorem 4.5).

2. Preliminaries

Throughout this paper, we assume that k is always an infinite field. Let A be a standard graded algebra over a field k, that is, A is a graded ring $\bigoplus_{i\geq 0} A_i$ satisfying $A_0=k$, $A=k[A_1]$ and $\dim_k A_1<\infty$. The *Hilbert function* of A is defined by $H(A,i)=\dim_k A_i$ for all $i\geq 0$, and the *Hilbert series* of A is defined by $F(A,\lambda)=\sum_{i\geq 0} H(A,i)\lambda^i \in \mathbf{Z}[[\lambda]]$. Then it is well known that we can write $F(A,\lambda)$ in the form

$$F(A,\lambda)=\frac{h_0+h_1\lambda+\cdots+h_c\lambda^c}{(1-\lambda)^d}$$

for certain integers $h_0,h_1,\ldots,h_c$ satisfying $\sum_{i=0}^{c} h_i\neq 0$ and $h_c\neq 0$, where d is the Krull-dimension of A. We put $\sigma(A)=c+1$. In particular, it follows that if A is Artinian, then $\sigma(A)=\operatorname{Min}\{i \mid A_i=(0)\}$.

First, we recall some basic facts on the Hilbert functions of finite sets of points in $\mathbf{P}^n$. Let X be a finite set of points in $\mathbf{P}^n$ and let $I(X)\subset R=k[x_0,x_1,\ldots,x_n]$ be the homogeneous ideal of X. The Hilbert function of X is defined by $H(X,i)=H(R/I(X),i)$ for all $i\geq 0$, the Hilbert series of X is defined by $F(X,\lambda)=F(R/I(X),\lambda)$ and put $\sigma(X)=\sigma(R/I(X))$. Now, since $A=R/I(X)=\bigoplus_{i\geq 0} A_i$ is a one-dimensional Cohen–Macaulay graded algebra over an infinite field, there exists an element $l\in A_1$ which is a non-zero divisor. Using this fact, it is easy to show the following (see [2], for example).

Lemma 2.1. *Let X be a finite set of points in $\mathbf{P}^n$.*

(1) $H(X,i)\leq H(X,i+1)$ *for all* $i\geq 0$.
(2) *If* $H(X,i+1)=H(X,i)$, *then* $H(X,i+2)=H(X,i+1)$.
(3) $H(X,i)=|X|$ *for all* $i\gg 0$, *where* $|X|$ *denotes the number of points in* X.
(4) $\sigma(X)=\operatorname{Min}\{i \mid H(X,i)=|X|\}+1$.

If $I(X)=(F_1,\ldots,F_n)$ for some $F_i\in R_{d_i}$ $(1\leq i\leq n)$, then X is said to be *complete intersection of type* $(d_1,\ldots,d_n)$. The following is also a well-known fact (see [2,11]).

Lemma 2.2. *Let X be a complete intersection of type $(d_1,\ldots,d_n)$.*

(1) $F(X,\lambda)=(\prod_{i=1}^{n}(1-\lambda^{d_i}))/((1-\lambda)^{n+1})$.
(2) $\sigma(X)=d_1+\cdots+d_n-(n-1)$.

Next, we recall some well-known facts on the graded Betti numbers of codimension three Gorenstein algebras.

When I is a Gorenstein homogeneous ideal of height three in $R=k[x_0,x_1,\dots,x_n]$, it is well known that a minimal graded free resolution of $A=R/I$ has the form

$$0\to R(-s)\to \bigoplus_{i=1}^{2m+1} R(-p_i)\to \bigoplus_{i=1}^{2m+1} R(-q_i)\to R(0)\to A\to 0,$$

where we always assume that $q_1\le\cdots\le q_{2m+1}$ and $p_1\ge\cdots\ge p_{2m+1}$. The set of integers

$$\{q_1,\dots,q_{2m+1};\,p_1,\dots,p_{2m+1};\,s\}$$

is uniquely determined by A, which we call the *graded Betti numbers* of A. In particular, we put $s(A)=s$. Furthermore, we define a new sequence $\{r_1,\dots,r_{2m+1}\}$ of integers, where

$$r_i=p_i-q_i \quad \text{for all } 1\le i\le 2m+1.$$

We call these integers the *diagonal degrees* of A (cf. [1, 3]).

It follows from [1, p. 466] that the diagonal degrees of A completely determine the graded Betti numbers of A, that is,

(BE1) $\quad s=\sum_{i=1}^{2m+1} r_i,$

(BE2) $\quad q_i=\frac{1}{2}(s-r_i)=\frac{1}{2}\sum_{j\ne i} r_j,$

(BE3) $\quad p_i=s-q_i=\frac{1}{2}(s+r_i).$

Furthermore, it is well known that the diagonal degrees $\{r_i\}$ of A satisfy the following three conditions (see [3, Proposition 3.1], for example):

(D1) $\quad r_1\ge r_2\ge\cdots\ge r_{2m+1},$

(D2) $\quad$ the integers r_i are all even or all odd,

(D3) $\quad r_1>0,\ r_2+r_{2m+1}>0,\ r_3+r_{2m}>0,\dots,r_{m+1}+r_{m+2}>0.$

Conversely, any sequence of integers satisfying the conditions (D1)–(D3) is the diagonal degrees of a height three Gorenstein ideal. This well-known fact follows, for example, from [3, Proposition 3.1; 5, Theorem 2.1; 8, Section 5].

3. The construction

We prepare the following notation and definitions to state the Main Theorem 3.3 of this section.

Let $R=k[x,y,z]$ be the homogeneous coordinate ring of $\mathbf{P}^2$. Here we consider the following finite sets of points which are in position of lattice points in $\mathbf{P}^2$.

Definition 3.1. (1) A finite set X of points in $\mathbf{P}^2$ is called a *basic configuration of type* (d,e) if there exist distinct elements b_j, c_j in k such that

$$I(X) = \left(\prod_{j=1}^{d}(x - b_j z),\ \prod_{j=1}^{e}(y - c_j z) \right).$$

We write $X = B(d,e)$. Obviously, $B(d,e)$ is complete intersection and $|B(d,e)| = de$.

(2) A finite set X of points in $\mathbf{P}^2$ is called a *pure configuration* if there exist finite basic configurations $B(d_1,e_1),\dots,B(d_m,e_m)$, where $e_1 > \cdots > e_m$, which satisfy the following three conditions:

(i) $B(d_i,e_i) \cap B(d_j,e_j) = \phi$ if $i \neq j$,

(ii) $X = B(d_1,e_1) \cup \cdots \cup B(d_m,e_m)$,

(iii) $\varphi(B(d_i,e_i)) \supset \varphi(B(d_{i+1},e_{i+1}))$ for all $1 \le i \le m-1$, where $\varphi : \mathbf{P}^2 \backslash \{(1,0,0)\} \to \mathbf{P}^1$ is the map defined by sending the point (x,y,z) to the point (y,z).

In this case, we write $X = \bigcup_{i=1}^{m} B(d_i,e_i)$.

Definition 3.2. Let $\{r_1,\dots,r_{2m+1}\}$ be a sequence of integers satisfying conditions (D1)–(D3). Then we define the following integers:

$$d_i = \tfrac{1}{2}(r_{m+2-i} + r_{m+1+i}) \quad \text{for all } 1 \le i \le m,$$

$$d_{m+1} = \tfrac{1}{2}(r_1 + r_{m+1}),$$

$$e_m = \tfrac{1}{2}(r_1 + r_{2m+1}),$$

$$e_i - e_{i+1} = \tfrac{1}{2}(r_{m+1-i} + r_{m+1+i}) \quad \text{for all } 1 \le i \le m-1,$$

$$e = e_1 \quad \text{and} \quad d = \sum_{i=1}^{m+1} d_i.$$

It follows from condition (D2) that all of d_i and e_i are integers. Furthermore, we can check from conditions (D1) and (D3) that

$$d_i > 0 \quad \text{for all } i, \text{ and } e_1 > e_2 > \cdots > e_m > 0.$$

Hence, there are a number of pairs

$$\left(X = \bigcup_{i=1}^{m} B(d_i,e_i),\ B = B(d,e) \right)$$

of pure and basic configurations such that $X \subset B$. For such pairs (X,B), we put

$$Y = \{P \in B \mid P \notin X\},$$

and we consider these pairs (X,Y). We call such pairs (X,Y) the *G-pairs* of $\{r_i\}$.

Theorem 3.3. *Let (X, Y) be a G-pair of a sequence $\{r_1, \ldots, r_{2m+1}\}$ of integers satisfying conditions* (D1)–(D3), *and put $A = R/I(X) + I(Y)$. Then the diagonal degrees of the Artinian Gorenstein algebra A are equal to the given integers r_i.*

In order to prove this theorem, we prepare a lemma.

Notation. Let $X = \bigcup_{i=1}^{m} B(d_i, e_i)$ be a pure configuration. Then obviously, there exist elements b_j, c_j in k such that

$$I(B(d_i, e_i)) = \left(\prod_{j=v_{i-1}+1}^{v_i} (x - b_j z), \prod_{j=1}^{e_i} (y - c_j z) \right),$$

where $v_0 = 0$ and $v_i = d_1 + \cdots + d_i$ for all $1 \le i \le m$. We put

$$g_i = \prod_{j=v_{i-1}+1}^{v_i} (x - b_j z) \quad \text{and} \quad h_i = \prod_{j=e_{i+1}+1}^{e_i} (y - c_j z)$$

for all $1 \le i \le m$, where $e_{m+1} = 0$. Note that $\deg g_i = d_i$ and $\deg h_i = e_i - e_{i+1}$ for all i. Furthermore, let $B = B(d, e)$ be a basic configuration such that $d > \sum_{i=1}^{m} d_i$, $e = e_1$ and $X \subset B$. Obviously, there exist elements b_j $(v_m + 1 \le j \le d)$ in k such that

$$I(B(d, e)) = \left(\prod_{j=1}^{d} (x - b_j z), \prod_{j=1}^{e} (y - c_j z) \right).$$

We put

$$g_{m+1} = \prod_{j=v_m+1}^{d} (x - b_j z) \quad \text{and} \quad d_{m+1} = d - \sum_{i=1}^{m} d_i.$$

In the following lemma, we describe a set of minimal generators of a height three Artinian Gorenstein ideal which is constructed as the sum of the ideals of two geometrically linked pure configurations in $\mathbf{P}^2$.

Lemma 3.4. *With the notation as above, we put $Y = \{P \in B \mid P \notin X\}$.*

(1) *$I(X)$ is minimally generated by $(m+1)$ maximal minors of the $m \times (m+1)$ matrix $U = (u_{ij})$ as follows:*

$$U = \begin{pmatrix} g_1 & h_1 & & & \mathbf{O} \\ & g_2 & h_2 & & \\ & & \ddots & \ddots & \\ \mathbf{O} & & & g_m & h_m \end{pmatrix}.$$

(2) *$I(X) + I(Y)$ is a Gorenstein ideal of height three, minimally generated by $(2m+1)$ pfaffians of the $(2m+1) \times (2m+1)$ alternating matrix $M = (f_{ij})$ as*

follows: *For* $i \le j$,

$$f_{ij} = \begin{cases} u_{it} & \text{if } 1 \le i \le m \text{ and } j = m+t \text{ for } 1 \le t \le m+1, \\ g_{m+1} & \text{if } i = m+1 \text{ and } j = 2m+1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (1) The set of all maximal minors of U is

$$B = \{h_1h_2\cdots h_m,\ g_1h_2\cdots h_m,\ g_1g_2h_3\cdots h_m, \ldots, g_1\cdots g_{m-1}h_m,\ g_1g_2\cdots g_m\}.$$

So we show that $I(X)$ is minimally generated by B. Let I be the ideal generated by B. We consider the monomial ideal J in $k[x,y]$ generated by the $(m+1)$ monomials $\{y^{e_1}, x^{v_1}y^{e_2}, x^{v_2}y^{e_3}, \ldots, x^{v_{m-1}}y^{e_m}, x^{v_m}\}$. Since $e_1 > e_2 > \cdots > e_m > 0$ and $0 < v_1 < v_2 < \cdots < v_m$, it is easy to show that J is minimally generated by the $(m+1)$ monomials above. Moreover, it follows by virtue of the proof of [4, Theorem 2.2] that I is a lifting of J (cf. [4, Definition 1.7] for the definition of "lifting"). Hence, I is the radical ideal which is minimally generated by B. And further, we can easily check that $X = \{P \in \mathbf{P}^2 \mid F(P) = 0 \text{ for all } F \in I\}$. Thus, we get $I = I(X)$.

(2) First of all, it follows from [10, Remarque 1.4; 6, Lemma 1.3] that $I(X)+I(Y)$ is a Gorenstein ideal of height three.

Next, we note that Y is also a pure configuration. Hence similarly, it follows from (1) that $I(Y)$ is minimally generated by

$$B' = \{h_1h_2\cdots h_m,\ h_1\cdots h_{m-1}g_{m+1},\ h_1\cdots h_{m-2}g_mg_{m+1}, \ldots,$$
$$h_1g_3\cdots g_{m+1},\ g_2g_3\cdots g_{m+1}\}.$$

Let F_i be the pfaffians of the alternating matrix obtained by eliminating the ith row and ith column from M for all $1 \le i \le 2m+1$. Then we can check that

$$F_1 = g_2g_3\cdots g_{m+1}, \qquad F_2 = h_1g_3\cdots g_{m+1}, \ldots, \qquad F_m = h_1\cdots h_{m-1}g_{m+1},$$
$$F_{m+1} = h_1h_2\cdots h_m,$$
$$F_{m+2} = g_1h_2\cdots h_m, \ldots, \qquad F_{2m} = g_1\cdots g_{m-1}h_m, \qquad F_{2m+1} = g_1g_2\cdots g_m.$$

Thus, $I(X)+I(Y)$ is generated by $B \cup B'$. So we check that $B \cup B'$ is a set of minimal generators of $I(X)+I(Y)$. We divide the proof of this claim into three cases.

Case 1: If $F_i \in (B \cup B' \setminus \{F_i\})R$ for some $1 \le i \le m$, then $F_i \in g_iR + h_iR$. Hence, taking a point Q such that $g_i(Q) = h_i(Q) = 0$, we get $F_i(Q) = 0$. But obviously, $F_i(P) \ne 0$ for all $P \in \mathbf{P}^2$ such that $g_i(P) = h_i(P) = 0$. This is a contradiction.

Case 2: If $F_{m+1} \in (B \cup B' \setminus \{F_{m+1}\})R$, then $F_{m+1} \in g_1R + g_{m+1}R$. Hence, taking a point Q such that $g_1(Q) = g_{m+1}(Q) = 0$, we get $F_{m+1}(Q) = 0$. But obviously, $F_{m+1}(P) \ne 0$ for all $P \in \mathbf{P}^2$ such that $g_1(P) = g_{m+1}(P) = 0$. This is a contradiction.

Case 3: If $F_{m+1+i} \in (B \cup B' \setminus \{F_{m+1+i}\})R$ for some $1 \le i \le m$, then $F_{m+1+i} \in g_{i+1}R + h_iR$. Hence, taking a point Q such that $g_{i+1}(Q) = h_i(Q) = 0$, we get $F_{m+1+i}(Q) = 0$. But obviously, $F_{m+1+i}(P) \ne 0$ for all $P \in \mathbf{P}^2$ such that $g_{i+1}(P) = h_i(P) = 0$. This is a contradiction. □

Proof of Theorem 3.3. We prove this theorem with the notation introduced above. From the proof of Lemma 3.4, the degrees of $\{F_i\}$ are as follows:

$$\deg F_1 = d_2 + d_3 + \cdots + d_{m+1}, \qquad \deg F_2 = (e_1 - e_2) + d_3 + \cdots + d_{m+1},$$
$$\deg F_3 = (e_1 - e_3) + d_4 + \cdots + d_{m+1}, \ldots, \qquad \deg F_m = (e_1 - e_m) + d_{m+1},$$
$$\deg F_{m+1} = e_1, \qquad \deg F_{m+2} = d_1 + e_2, \qquad \deg F_{m+3} = d_1 + d_2 + e_3, \ldots,$$
$$\deg F_{2m} = d_1 + \cdots + d_{m-1} + e_m, \qquad \deg F_{2m+1} = d_1 + d_2 + \cdots + d_m.$$

For convenience, we put

$$G_i = \begin{cases} F_{2m+2-i} & \text{for all } 1 \le i \le m+1, \\ F_{i-(m+1)} & \text{for all } m+2 \le i \le 2m+1. \end{cases}$$

Hence, from the definitions of d_i and e_i, we can easily check that

$$\deg G_i = \frac{1}{2}\sum_{j \ne i} r_j = \frac{1}{2}(r_1 + \cdots + r_{i-1} + r_{i+1} + \cdots + r_{2m+1})$$

for all $1 \le i \le 2m+1$. Thus, from condition (D1) of sequence $\{r_i\}$,

$$\deg G_1 \le \deg G_2 \le \cdots \le \deg G_{2m+1}.$$

Next, we show that $s(A) = \sum_{i=1}^{2m+1} r_i$. Note that $s(A) = \sigma(A) + 2$ (this follows, for example, from [3, p. 369]). Furthermore, it follows from [6, Theorem 2.1(3)] and Lemma 2.2(2) that

$$\sigma(A) = \sigma(X \cup Y) - 1 = \sigma(B(d,e)) - 1 = d + e - 2.$$

Hence, we get

$$s(A) = d + e = \left\{\sum_{i=1}^{m+1} d_i\right\} + e_1 = \left\{\sum_{i=1}^{m+1} d_i\right\} + \left\{\sum_{i=1}^{m-1}(e_i - e_{i+1})\right\} + e_m = \sum_{i=1}^{2m+1} r_i.$$

Now, let $\{r'_1, \ldots, r'_{2m+1}\}$ be the diagonal degrees of A. Then by noting that $\deg G_1 \le \deg G_2 \le \cdots \le \deg G_{2m+1}$, it follows from conditions (BE1) and (BE2) that

$$r'_i = s(A) - 2\deg G_i, \quad \text{i.e.,} \quad r'_i = \sum_{j=1}^{2m+1} r_j - \sum_{j \ne i} r_j = r_i.$$

This completes the proof. □

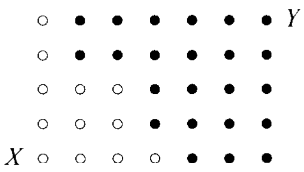
Example 3.5. Diesel described in [3, Example 3.7] all the possible diagonal degrees among all Artinian Gorenstein algebras with the Hilbert function $T = (1,3,6,10,12,12,10,6,3,1,0,\ldots)$, i.e., all the sequences of integers satisfying conditions (D1)–(D3) which determine T:

$$\{4,4,4\}; \ \{4,4,4,2,-2\}; \ \{4,4,4,0,0\}; \ \{4,4,4,2,2,-2,-2\};$$
$$\{4,4,4,2,0,0,-2\}; \quad \{4,4,4,2,2,0,0,-2,-2\}.$$

Here using our construction, for example, we give an example of an Artinian Gorenstein algebra with the diagonal degrees $\{4,4,4,2,0,0,-2\}$. We put, as in Definition 3.2,

$$d_1=1,\quad d_2=2,\quad d_3=1,\quad d_4=3,$$
$$e_1=5,\quad e_2=3,\quad e_3=1,\quad d=7,\quad \text{and}\quad e=5.$$

And, as a G-pair (X, Y) of $\{4,4,4,2,0,0,-2\}$, we take the following two pure configurations $X=B(1,5)\cup B(2,3)\cup B(1,1)$ and $Y=B(3,5)\cup B(1,4)\cup B(2,2)$ such that $X\cup Y=B(7,5)$:



Then it follows from Theorem 3.3 that $A=R/I(X)+I(Y)$ is an Artinian Gorenstein algebra with the diagonal degrees $\{4,4,4,2,0,0,-2\}$.

4. An application

Definition 4.1 (cf. Diesel [3]). Let $\{r_1,\ldots,r_{2m+1}\}$ be a sequence of integers satisfying the conditions (D1)–(D3). We say that $\{r_i\}$ is saturated if

$$r_i+r_{2m+3-i}=2 \quad \text{for all} \quad 2\le i\le m+1.$$

Definition 4.2 (cf. Watanabe [12]). Let $A=\bigoplus_{i=0}^{c} A_i$ be an Artinian graded algebra. We say that A has the *weak Stanley property* if

(i) the Hilbert function of A is unimodal, i.e., there exists an integer j such that $H(A,0)\le H(A,1)\le\cdots\le H(A,j)\ge H(A,j+1)\ge\cdots\ge H(A,c)$, and

(ii) there exists an element $l\in A_1$ such that the multiplication $l: A_i\to A_{i+1}$ defined by $f\mapsto lf$ is either injective or surjective for all $i\ge 0$.

In this case, we say that the pair (A,l) has the weak Stanley property.

Theorem 4.3. *Let (X,Y) be a G-pair of a saturated sequence $\{r_1,\ldots,r_{2m+1}\}$, and put $A=R/I(X)+I(Y)$. Furthermore, put $a=\sigma(X)-1$, $b=\sigma(X\cup Y)-\sigma(X)-1$ and $c=\sigma(X\cup Y)-2$. Then A has the weak Stanley property and the Hilbert function of A is recovered from the Hilbert function of X as follows:*

$$H(A,i)=\begin{cases} H(X,i) & \text{for all } 0\le i\le a-1,\\ |X| & \text{for all } a\le i\le b,\\ H(X,c-i) & \text{for all } b+1\le i\le c,\end{cases}$$

i.e., $H(A)=(1,h_1,\ldots,h_{a-1},|X|,\ldots,|X|,h_{a-1},\ldots,h_1,1,0,\ldots)$, where $h_i=H(X,i)$.

We need the following lemma to prove Theorem 4.3.

Lemma 4.4. *Let $X = \bigcup_{i=1}^{m} B(d_i, e_i)$ be a pure configuration.*

(1) $$F(X,\lambda) = \sum_{i=1}^{m} \lambda^{v_{i-1}} \frac{(1-\lambda^{d_i})(1-\lambda^{e_i})}{(1-\lambda)^3},$$

where $v_0 = 0$ and $v_i = d_1 + \cdots + d_i$.

(2) $$\sigma(X) = \mathrm{Max}\{e_i + v_i - 1 \mid 1 \le i \le m\}.$$

Proof. (1) We use induction on m. For the case of $m = 1$, our assertion follows from Lemma 2.2(1). Let $m > 1$. It follows from Lemma 3.4(1) that

$$I\left(\bigcup_{i=1}^{m-1} B(d_i, e_i)\right) = (h_1 \cdots h_m, g_1 h_2 \cdots h_m, g_1 g_2 h_3 \cdots h_m, \ldots, g_1 \cdots g_{m-2} h_{m-1} h_m, g_1 \cdots g_{m-1}).$$

Hence, we have

$$I\left(\bigcup_{i=1}^{m-1} B(d_i, e_i)\right) + I(B(d_m, e_m)) = (g_1 \cdots g_{m-1}, g_m, h_m).$$

Thus, we obtain the following exact sequence:

$$0 \to R/I(X) \to R/I\left(\bigcup_{i=1}^{m-1} B(d_i, e_i)\right) \oplus R/I(B(d_m, e_m)) \to R/(g_1 \cdots g_{m-1}, g_m, h_m) \to 0.$$

Therefore, we get

$$F(X,\lambda) = F\left(R/I\left(\bigcup_{i=1}^{m-1} B(d_i, e_i)\right), \lambda\right) + F(R/I(B(d_m, e_m)), \lambda) - F(R/(g_1 \cdots g_{m-1}, g_m, h_m), \lambda).$$

On the other hand, by the assumption of induction,

$$F\left(R/I\left(\bigcup_{i=1}^{m-1} B(d_i, e_i)\right), \lambda\right) = \sum_{i=1}^{m-1} \lambda^{v_{i-1}} \frac{(1-\lambda^{d_i})(1-\lambda^{e_i})}{(1-\lambda)^3}.$$

Also since $\{P \in \mathbf{P}^2 \mid g_1 \cdots g_{m-1}(P) = g_m(P) = h_m(P) = 0\} = \phi$, it follows that $\{g_1 \cdots g_{m-1}, g_m, h_m\}$ is a homogeneous regular sequence. Hence by using Lemma 2.2(1), it follows that

$$\begin{aligned} &F(R/I(B(d_m, e_m)), \lambda) - F(R/(g_1 \cdots g_{m-1}, g_m, h_m), \lambda) \\ &= \frac{(1-\lambda^{d_m})(1-\lambda^{e_m})}{(1-\lambda)^3} - \frac{(1-\lambda^{v_{m-1}})(1-\lambda^{d_m})(1-\lambda^{e_m})}{(1-\lambda)^3} \\ &= \frac{\lambda^{v_{m-1}}(1-\lambda^{d_m})(1-\lambda^{e_m})}{(1-\lambda)^3}. \end{aligned}$$

Thus, we get the equality of (1).

(2) It follows from (1) that

$$F(X,\lambda)=\sum_{i=1}^{m}\lambda^{v_{i-1}}F(B(d_i,e_i),\lambda),\quad \text{i.e., } H(X,j)=\sum_{i=1}^{m}H(B(d_i,e_i),j-v_{i-1}).$$

Here we put

$$\tau(i)=\mathrm{Min}\{j\mid H(B(d_i,e_i),j-v_{i-1})=|B(d_i,e_i)|\}\quad \text{for all } 1\le i\le m.$$

Then from Lemmas 2.1(4) and 2.2(2), we see that

$$\tau(i)-v_{i-1}=\sigma(B(d_i,e_i))-1=d_i+e_i-2,\quad \text{i.e., } \tau(i)=e_i+v_i-2.$$

Thus, we can check that

$$\mathrm{Min}\left\{j\mid H(X,j)=\sum_{i=1}^{m}|B(d_i,e_i)|\right\}=\mathrm{Max}\{e_i+v_i-2\mid 1\le i\le m\}.$$

Therefore, from Lemma 2.1(4), we get the equality of (2). □

Proof of Theorem 4.3. If $2\sigma(X)\le\sigma(X\cup Y)$, then our assertion follows from [7, Lemmas 3.1 and 3.2]. So we show that $2\sigma(X)\le\sigma(X\cup Y)$. Since $\{r_i\}$ is saturated, we have $d_i=1$ for all $1\le i\le m$, i.e., $v_i=i$. Hence from $e_1>e_2>\cdots>e_m$, it follows that $e_i+v_i-1\ge e_{i+1}+v_{i+1}-1$. Thus, from Lemma 4.4(2) and the definition of e_1, we have

$$\sigma(X)=e_1+v_1-1=e_1=\frac{1}{2}\sum_{i\ne m+1}r_i.$$

Furthermore, from Lemma 2.2(2) and Definition 3.2, we have

$$\sigma(X\cup Y)=d+e-1=\left(\sum_{i=1}^{m-1}d_i\right)+e_1-1=\left(\sum_{i=1}^{2m+1}r_i\right)-1.$$

Also, we see that $r_{m+1}>0$, because $r_{m+1}\ge r_{m-2}$ and $r_{m+1}+r_{m+2}=2$. Thus, it follows that $\sigma(X\cup Y)-2\sigma(X)=r_{m+1}-1\ge 0$. □

Theorem 4.5 (cf. Stanley [11, Theorem 4.2]). *Let $h=(h_0,h_1,\ldots,h_c,0,\ldots)$ be a sequence of non-negative integers satisfying $h_1=3$. Then the following conditions are equivalent*:

(a) *there exists an Artinian Gorenstein graded algebra with the Hilbert function h*;

(b) *$h_i=h_{c-i}$ for all $0\le i\le[c/2]$ and the sequence $(h_0,h_1-h_0,h_2-h_1,\ldots,h_{[c/2]}-h_{[c/2]-1},0,\ldots)$ is the Hilbert function of an Artinian graded algebra.*

Proof. (a) ⇒ (b): It follows from [3, Theorem 3.2] that there is a unique saturated sequence $\{r_i\}$ which determine h. We take a G-pair (X,Y) of $\{r_i\}$, and put

$A = R/I(X) + I(Y)$. We note that the Hilbert function of A is equal to h. Hence, it follows from Theorem 4.3 that $h_i = h_{c-i}$ for all $0 \leq i \leq [c/2]$. Furthermore, it follows from Theorem 4.3 that (A, l) has the weak Stanley property, that is, $(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{[c/2]} - h_{[c/2]-1}, 0, \ldots)$ is the Hilbert function of A/lA.

(b) $\Rightarrow$ (a) follows, for example, from [6, Theorem 3.3]. □

Remark 4.6. In the proof of Theorem 4.5, we give an algebraic explanation of what is behind Stanley's formulation (in terms of first difference) for Hilbert function. That is, for any Artinian Gorenstein algebra of codimension three, there exists an Artinian Gorenstein algebra with the weak Stanley property which has the same Hilbert function. Therefore, it is natural to ask, in view of Stanley's formulation, a question whether every Artinian Gorenstein algebra of codimension three has the weak Stanley property (cf. [9,12,13]).

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